Differences of Reversed Squares

A mathematician was bored during a long train ride from Upstate New York to New York City. Having a little notebook and a pencil always handy, she began to scribble the following:

$$21^2 - 12^2 = 441 - 144 = 297$$
$$31^2 - 13^2 = 961 - 169 = 792$$

wow!

$$41^2 - 14^2 = ?$$
$$342^2 - 243^2 = ?$$

Hmmm...

(1) What is the mathematician investigating here?

(2) Can you find patterns when you continue her investigation for 2-digit numbers? For 3-digit numbers? For numbers of any amount of digits?

(3) The differences above (297 and 792) are divisible by 99. What about the differences of reversed squares for any 2-digit number? What about 3-digit numbers? What about numbers with any amount of digits?

Prepare a presentation of your work in writing. Imagine you would be sharing your discoveries at a conference for mathematicians at which you are required to prove all your claims.

You might want to review some basic algebra with respect to differences of squares.

Besides the questions above, what interesting questions could you make up for the bored mathematician as an extension to this problem?

Differences of Reversed Squares

Suggested Grade Span

9–12

Task

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Context

This problem is actually a true story. I was bored on a train ride and did play with differences of reversed squares. The problem is posed in the authentic context of a mathematician. I feel students place themselves in such a role. So in this case, mathematics forms a context for itself. Another point is that in the 1989 NCTM Standards, Number Theory is not officially recommended as a strand continuing in high school. I hope that this task will demonstrate that there is value in working on this.

What This Task Accomplishes

This task employs algebra to show one's findings in general. Rules of divisibility must be employed in their algebraic forms in order to generalize. Also the student must represent the place value system algebraically. Proof is required.

What the Student Will Do

Students will be able to experiment and look for patterns, develop conjectures about numbers, test them and prove them using the algebraic representation of the place value system.

Time Required for Task

This problem is best assigned independently over a period of time. Some class time (about 40 minutes) is recommended to start up the problem and to debrief the work at the end. Time for the debriefing will depend on how you structure it.

Teaching Tips

Many students will want to investigate the numbers first. The use of a calculator or computer spreadsheet is highly recommended. During this work students often come to understand the problem and are convinced that a difference of reversed squares is always divisible by 99. Some students will be satisfied with this. They will say such things as "I have tried it for random numbers and it always works." One way to counter this is to ask them if they think it works for a number consisting of 100 digits. Ask them to show you. Their calculators cannot do this, of course. To do pencil and paper division would take them hours.

You may need to help students to represent a number in its algebraic form consistent with the place value system. For larger numbers, scientific notation for the place value makes the algebra more manageable. You may also need to review how to factor the difference of two squares. Find out from your students what other previous knowledge they will need to recall. You may need to review or establish methods of proof, particularly mathematical induction if you wish to have students get to the most general case.

Suggested Materials

A scientific calculator or a spreadsheet is handy for exploration.

Possible Solutions

Differences of squares in which the order of the digits of the first number are reversed to get the second number are always divisible by 99.

Examples: 212 - 122 = 441 - 144 = 297 = 3 x 99
312 - 132 = 961 - 169 = 792 = 8 x 99
412 - 142 = 1681 - 196 = 1485 = 15 x 99

The case of two digits is proved as follows:
$[10a + b]^2 - [10b + a]^2 = [10a + b + 10b + a] [10a + b - 10b - a] =$
$[11a + 11b] [9a - 9b] = 99[a^2 - b^2]$
Note how one factor is divisible by 11 and the other by 9.

The case for differences of three-digit reversed squares goes similarly:
$[100a + 10b + c]^2 - [100c + 10b + a]^2 =$
$[101a + 20b + 101c] [99a - 99c] =$
$99 [101a + 20b + 101c] [a - c]$
Note how one of the factors is divisible by 99.

One could continue to prove cases ad infinitum, but since this is a never-ending job we need a better way. Let’s consider the following conjecture:

It appears that if one continues this pattern of proof that the difference of reversed squares, $x^2 - xr^2$, has either one of its factors divisible by 11 and the other by nine or one of its factors divisible by 99 and the other factor relatively prime. This means that $x^2 - xr^2$ is divisible by $(10 + 1) (10 - 1) = 10^2 - 1$. It can be shown that this works in bases other than 10 as well. It is true in general that for any base b, $xb^2 - xb,r^2$ is divisible by $b^2 - 1$.

The problem can be extended to differences of reversed cubes as well: $x^3 - xr^3$ is divisible by 27. The binomial theorem is quite useful here. Further generalizations of this problem are possible, but may fall beyond the scope of high school.

Task-Specific Assessment Notes

General Notes

In the pilot task the title was “differences of reverted squares.” It was pointed out that the word reverted should be replaced by reversed. This correction has been made, but in the student work you will see a mix of these two words. We apologize for this and we hope this will not be problematic.

Novice
The student recognizes differences of two squares and does some calculations. It is not clear what the student has figured out. There seems to be some understanding and an attempt to look for patterns.

Apprentice
The student appears to understand the reversing. The student explains attempts to find a continuing pattern of reversing and states that it could not be found. In communicating, the student does not distinguish between numbers and digits, making the work hard to follow. In an answer to question three the student draws a conclusion without proof.

Practitioner
Although this paper lacks clear communication about the work, it is possible to follow the student's thinking. It appears that this student becomes convinced that division by 99 works. Then the student sets out to prove this for two-digit numbers and three-digit numbers using appropriate algebra to represent the place value system.

Expert
An Expert paper could not be identified. It would require a proof using mathematical induction or one with another method.

Novice

1. The difference of two squares
2. yes.

$21^2 - 12^2 = 297 \div 99 = 3$
$31^2 - 13^2 = 792 \div 99 = 8$
$41^2 - 14^2 = 1485 \div 99 = 15$
3. $51^2 - 15^2 = 2{,}376 \div 99 = 24$
$61^2 - 16^2 = 3465 \div 99 = 35$
$71^2 - 17^2 = 4252 \div 99 = 48$

5
7
9
11
13

The student appears to look for a pattern. No explanations are given.

Apprentice

1. The mathematician wants to see if she can use any numbers and square it, then subtract the reverse of that number squared, to see if that answer is also reversed.

Ex. $21^2 = 441$
$12^2 = 144$ } the answer is reverse just like the numbers that was reverse from greatest possible to smallest.

$21^2 - 12^2 = 441 - 144 = 297$
$31^2 - 13^2 = 961 - 196 = 792$ } when using the different #s the mathmatician want to see the pattern of the errors between the two equations and she wants to see if she can use any #s to get the reverse.

The student understands the reversing.

The student is not clear about what is not working.

2. I can not find the pattern between 2#s, 3#s, or any amount of #s. I tried different amount of #s but it didnt work. Such as $51^2 - 15^2$, $61^2 - 16^2$. There wasn't any relationshyps between the two. I also tried the $10^2 - 01^2$ and $11^2 - 11^2$ it still didnt work. The reason I used 10 5 11 because I notice that when you square 2#s there should be 3# answer meaning that if I square the 2nd # Their should be an answer w/ only one#.

Because communication is unclear, it is difficult to follow. The symbol "#" seems to be used with different meanings.

Apprentice

3. The differences of reverted squares for any 2 digit # are also divisible by 99. such as 81^2-18^2, 51^2-15^2, 61^2-16^2

The differences of reverted square for any 3 digit or any amount of digits are also divisible by 99 as long as they revert the differences squares.
EX. $562^2 - 265^2$ is divisible by 99.

The student restates the question but does not show any work or conjecture or proof.

Conclusion:

In conclusion, I think the mathmatician might figure that the diffaunce of reverted squares for any amount of numbers are always divisible by 99. I'm not sure what's the patterns besides that the numbers were squared.

I think the mathmatician can try to find the two equations that will result with reverted answers.

The student is still focused on the symmetry of the first two examples.

question for mathmatician.

~~Does~~ Can the differences of any reverted squares be divisible by any # besides 99?

The student poses a viable question.

Practitioner

$A^2 - B^2 \quad (a+b)(a-b)$

$(x+10n)^2 - (10x+n)^2 \qquad (\quad - \quad)(\quad)$

45

all reverted squares ÷ 99

Her pattern is

$11^2 - 11^2 = 0$
$21^2 - 12^2 = 297 \div 99 = 3$ +5
$31^2 - 13^2 = 792 \div 99 = 8$ +2 +7
$41^2 - 14^2 = 1485 \div 99 = 15$ +2 +9
$51^2 - 15^2 = 2376 \div 99 = 24$ +2 +13
$61^2 - 16^2 \quad 3465 \div 99 = 35$ +2 +15
$71^2 - 17^2 \quad 4752 \div 99 = 49$ +2 +17
$81^2 \; 18^2 \quad 6237 \div 99 = 63$ +2 +19
$91^2 - 19 = 7920 \div 99 = 80$ +2 +21
$100^2 - 1^2 = 9999 \div 99 = 101$
$98542^2 - 24589^2 = 9105906843 \div 99 = 91978857$

124

$42^2 - 24^2 = 99 = $
$32^2 - 23^2 \div 99$ 5
$43^2 - 34^2 - 99 =$ 7
$54^2 - 45^2 =$ 9

The algebraic notation is appropriate for this problem.

A conjecture is made.

The student shows examples and seems to discover a pattern.

Practitioner

$$(10t+u)^2-(10u+1t)^2$$

$$100t^2+\cancel{20tu}+u^2 \quad +(100u^2+\cancel{20ut}+t^2)$$

$$99t^2-99u^2=\underline{\underline{99(t^2-u^2)}}$$

$$(100h+10t+u)^2-(100u+10t+h)^2$$

$$\begin{array}{rrrr} & 100u & 10t & 1h \\ & 100u & \cancel{10t} & \cancel{1h} \\ \hline & 100uh & 10tu & h^2 \\ & & 10th & 0 \\ 10000h & 1000tu\;100t^2 & & \\ 10000u^2 & 1000tu & 100uh & 0 \end{array}$$

$$\begin{array}{rrrr} & 100h & +10t & u \\ & 100h & +\cancel{10t} & \cancel{u} \\ \hline & 100h & 10tu & u^2 \\ 1000ht & 0 & 10tu & 0 \\ 10000h & 1000ht\;100hu & +0 & +0 \end{array}$$

$$\begin{array}{l} 10000h^2+2000ht+\cancel{200hu}+20tu+u^2+100t^2 \\ -\quad h^2 \quad 20ht \quad \cancel{200uh} \quad 2000tu-10000u^2 \quad 100t^2 \end{array}$$

$$(9999h^2+1980ht-1980tu-9999u^2)$$

$$99(101h^2+20ht-20tu-101u^2)$$

Correct proofs are given for 2-digit and 3-digit numbers.

Expert

Algebraic proof is given and clearly communicated. Numbers are used as illustrations.

Differences of Reversed Squares

examples (2 digits):

$11^2 - 11^2 = 121 - 121 = 0\ (99 \times 0)$

$21^2 - 12^2 = 441 - 144 = 297\ (99 \times 3)$

$31^2 - 13^2 = 961 - 169 = 792\ (99 \times 8)$

$41^2 - 14^2 = 1681 - 196 = 1485\ (99 \times 15)$

$51^2 - 15^2 = 2601 - 225 = 2376\ (99 \times 24)$

$61^2 - 16^2 = 3721 - 256 = 3465\ (99 \times 35)$

$71^2 - 17^2 = 5041 - 289 = 4752\ (99 \times 48)$

$81^2 - 18^2 = 6561 - 324 = 6237\ (99 \times 63)$

$91^2 - 19^2 = 8281 - 361 = 7920\ (99 \times 80)$

$100^2 - 1^2 = 10{,}000 - 1 = 9999\ (99 \times 101)$

$62^2 - 26^2 = 3844 - 676 = 3168\ (99 \times 32)$

$74^2 - 47^2 = 5476 - 2209 = 3267\ (99 \times 53)$

etc.

$A^2 - B^2 = (10a+b)^2 - (10b+a)^2$

$= (100a^2 + 20ab + b^2) - (100b^2 + 20ab + a^2)$

$= 99a^2 - 99b^2$

$= 99(a^2 - b^2)$

Therefore, all differences of reversed squares, for 2 digits, are divisible by "99" and the difference of the squares of their digits. $A^2\ B^2$ could also be reduced to $(A-B)(A+B)$, and one would receive the same final product.

examples (3 digits):

$111^2 - 111^2 = 12321 - 12321 = 0\ (99 \times 0)$

$211^2 - 112^2 = 44521 - 12544 = 31{,}977\ (99 \times 323)$

$311^2 - 113^2 = 96{,}721 - 12{,}769 = 83{,}952\ (99 \times 848)$

$411^2 - 114^2 = 168{,}921 - 12{,}996 = 155{,}925\ (99 \times 1575)$

$A^2 - B^2 = (100a + 10b + c)^2$

$(100c + 10b + a)^2$

$= (10{,}000a^2 + 2000ab + 20bc$

$\cancel{100b^2} + \cancel{2000ac} + c^2)$

$10{,}000c^2 + 2000bc + 20$

$\cancel{100b^2} + \cancel{200ac} + a^2)$

$(100a + 10bc + 100c + 10a)$

(cont.)

$A^2 - B^2 = 9999a^2 + 1980ab - 1980bc - 9999c^2 \times (100a + 10b + c - 100a + 10b + a)$

$= 99(101a^2 + 20ab - 20bc - 101c^2) = (99a - 99c)(101a + 20b + 101c)$

Therefore, all differences of reversed squares, for 3-digit numbers, are divisible by "99" and $(101a^2 + 20ab - 20bc - 101c^2)$

Then, move on to testing numbers with "n+1" digits: $A^2 - B^2 = (A-B)(A+B)$

$(10^n x_n + 10^{n-1} x_{n-1} + \ldots + 10x_1 + x_0)^2 - (10^n x_0 + 10^{n-1} x_1 + \ldots + 10x_{n-1} + x_n)^2$

$(10^n x_n + 10^{n-1} x_{n-1} + \ldots + 10x_1 + x_0 - 10^n x_0 - 10^{n-1} x_1 - \ldots - 10x_{n-1} - x_n) \times$

$(10^n x_n + 10^{n-1} x_{n-1} + \ldots + 10x_1 + x_0 + 10^n x_0 + 10^{n-1} x_1 + \ldots + 10x_{n-1} + x_n)$

The student makes an attempt to extend the problem to numbers with any amount of digits and uses scientific notation to represent the place value system.

Expert

$((10^n-1)x_n + (10^{n-1}-10)x_{n-1} + \cdots + (10^1-10^{n-1})x_1 + (10^0-10^n)x_0)$

$((10^n+1)x_n + (10^{n-1}+10)x_{n-1} + \cdots + (10^1+10^{n-1})x_1 + (10^0+10^n)x_0)$

when "n" is even:

(a): $n=4$ (a 5-digit number)

$(9999x_4 + 990x_3 - 990x_1 - 9999x_0)(10001x_4 + 1010x_3 + 200x_2 + 1010x_1 + 10001x_0)$

$99(101x_4 + 10x_3 - 10x_1 - 101x_0)(10001x_4 + 1010x_3 + 200x_2 + 1010x_1 + 10001x_0)$

Therefore, when "n" is even, and there are an odd number of digits in the number, it is divisible by "99."

when "n" is odd:

(ex): $n=3$ (a 4-digit number)

$(999x_3 + 90x_2 - 90x_1 - 999x_0)(1001x_3 + 110x_2 + 110x_1 + 1001x_0)$

$9(111x_3 + 10x_2 - 10x_1 - 111x_0) \cdot 11(91x_3 + 10x_2 + 10x_1 + 91x_0)$

Therefore, when "n" is odd, and there are an even number of digits in the number, it is also divisible by "99."

The student proves two more cases and makes a statement about odd and even amounts of digits in given numbers.

No proof is given for the general case.